A NOTE ON NON-ROBBA p-ADIC DIFFERENTIAL EQUATIONS

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Abstract. Let $\mathcal{M}$ be a differential module, whose coefficients are analytic elements on an open annulus I ($\subset \mathbb{R}_{>0}$) in a valued field, complete and algebraically closed of inequal characteristic, and let $R(\mathcal{M}, r)$ be the radius of convergence of its solutions in the neighbourhood of the generic point t_r of absolute value r, with $r \in I$. Assume that $R(\mathcal{M}, r) < r$ on I and, in the logarithmic coordinates, the function $r \longrightarrow R(\mathcal{M}, r)$ has only one slope on I. In this paper, we prove that for any $r \in I$, all the solutions of $\mathcal{M}$ in the neighborhood of t_r are analytic and bounded in the disk $D(t_r, R(\mathcal{M}, r)^-)$.

1. Notations and Preliminaries

Let p be a prime number, $\mathbb{Q}_p$ the completion of the field of rational numbers for the p-adic absolute value $|.|$, $\mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$, and Ω_p a p-adic complete and algebraically closed field containing $\mathbb{C}_p$ such that its value group is $\mathbb{R}_{\geq 0}$ and the residue class field is strictly transcendental over $\mathbb{F}_{p^\infty}$. For any positive real r, t_r will denote a generic point of Ω such that $|t_r| = r$. Let I be a bounded interval in $\mathbb{R}_{>0}$. We denote by $\mathcal{A}(I)$ the ring of analytic functions, on the annuli $\mathcal{C}(I) := \{a \in \Omega_p \mid |a| \in I\}$, $\mathcal{A}(I) = \left\{ \sum_{n\in\mathbb{Z}} a_n x^n \in \mathbb{C}_p[[x, 1/x]] \;\middle|\; \lim_{n\to\mp\infty} |a_n| r^n = 0, \forall\, r \in I \right\}$, and by $\mathcal{H}(I)$ the completion of the ring of rational fractions f of $\mathbb{C}_p(x)$ having no pole in $\mathcal{C}(I)$ with respect to the norm $\|f\|_I := \sup_{r\in I} |f(t_r)|$. It is well known that $\mathcal{H}(I) \subseteq \mathcal{A}(I)$, with equality if I is closed. We define, for any $r \in I$, the absolute value $|.|_r$ over $\mathcal{A}(I)$ by $\left| \sum_{n\in\mathbb{Z}} a_n x^n \right|_r = \sup_{n\in\mathbb{Z}} |a_n| r^n$.

Let $R(I)$ denotes $\mathcal{A}(I)$ or $\mathcal{H}(I)$. A free $R(I)$-module $\mathcal{M}$ of finite rank μ is said to be $R(I)$-differential module if it is equipped with a $R(I)$-linear map $D : \mathcal{M} \to \mathcal{M}$ such that $D(am) = \partial(a)m + aD(m)$ for any $a \in R(I)$ and any $m \in \mathcal{M}$ where $\partial = d/dx$. To each $R(I)$-basis $\{e_i\}_{1\leq i\leq\mu}$ of $\mathcal{M}$ over $R(I)$ corresponds a matrix $G = (G_{ij}) \in M_\mu(R(I))$ satisfying $D(e_i) = \sum_{j=1}^{\mu} G_{ij} e_j$, called the matrix of ∂ with respect to the $R(I)$-basis $\{e_i\}_{1\leq i\leq\mu}$ or simply an associated matrix to $\mathcal{M}$, together with a differential system $\partial X = GX$ where X denotes a column vector $\mu \times 1$ or $\mu \times \mu$ matrix (see [2], [3]). If $G' \in M_\mu(R(I))$ is the matrix of ∂ with

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respect to another $R(I)$-basis $\{e'_i\}_{1\leq i\leq \mu}$ of $\mathcal{M}$ and if $H=(H_{ij})\in \mathrm{GL}_\mu(R(I))$ is the change of basis matrix defined by $e'_i=\sum_{i=1}^{\mu} H_{ij}e_i$ for all $1\leq i\leq \mu$, it is known that:

- the matrices G and G' are related by the formula $G'=HGH^{-1}+\partial(H)H^{-1}$. The matrix $HGH^{-1}+\partial(H)H^{-1}$ is denoted $H[G]$.
- if Y is a solution matrix for the system $\frac{d}{dx}X=GX$ with coefficients in a differential field extension of $R(I)$, then the matrix HY is a solution matrix for $\frac{d}{dx}X=H[G]X$.

Generic radius of convergence. Let $\mathcal{M}$ be an $R(I)$-differential module of rank μ, $G=(G_{ij})\in \mathrm{M}_\mu(R(I))$ an associated matrix to $\mathcal{M}$, $(G_n)_n$ a sequence of matrices defined by

$$G_0=\mathbb{I}_\mu \quad \text{and} \quad G_{n+1}=\partial(G_n)+G_nG,$$

and $||G||_r=\max_{ij}|G_{ij}|_r$ be the norm of G associated to the absolute value $|.|_r$. For any $r\in I$, the quantity $R(\mathcal{M},r)=\min(r,\liminf_{n\to\infty}||G_n||_r^{-1/n})$ represents the radius of convergence in the generic disc $D(t_r,r^-)$ of the solution matrix

$$\mathcal{U}_{G,t_r}(x)=\sum_{n\geq 0}\frac{G_n(t_r)}{n!}(x-t_r)^n$$

of the system $\frac{d}{dx}X=GX$ with $X(t_r)=\mathbb{I}_\mu$. We know that the function $r\to R(\mathcal{M},r)$ is independent of the choice of basis and the ring $R(I)$ [3, Proposition 1.3], and the graph of the map $\rho\mapsto \log\circ R(\mathcal{M},\exp(\rho))$, on any closed subinterval of I, is a concave polygon with rational slopes [5, Theorem 2]. This graph is called the generic polygon of the convergence of $\mathcal{M}$. The system $\partial X=GX$ is said to have an analytic and bounded solution in the disk $D(t_r,R(\mathcal{M},r)^-)$ if

$$\sup_{n\geq 0}\left|\left|\frac{G_n}{n!}\right|\right|_r R(\mathcal{M},r)^n<\infty.$$

The $R(I)$-differential module $\mathcal{M}$ is said to be non-Robba if $R(\mathcal{M},r)<r$ for all $r\in I$.

Frobenius. Let $\varphi:\mathcal{C}(I)\to\mathcal{C}(I^p)$ be the Frobenius ramification $x\mapsto x^p$, where I^p is the image of I by the map $x\mapsto x^p$. A $R(I^p)$-differential module $\mathcal{N}$ is said to be a Frobenius antecedent of an $R(I)$-differential module $\mathcal{M}$ if $\mathcal{M}$ is isomorphic to the inverse image $\varphi^*\mathcal{N}$ of $\mathcal{N}$. In other words, if there exists a matrix $F\in \mathrm{M}_\mu(R(I^p))$ of the derivation d/dz (where $z=x^p$) in some $R(I^p)$-basis of $\mathcal{N}$ such that $px^{p-1}F(x^p)$ is a matrix of d/dx in some $R(I)$-basis of $\mathcal{M}$. The existence of such a Frobenius antecedent depends of the values of the function $r\mapsto R(\mathcal{M},r)$. Recall the Frobenius structure theorem of Christol-Mebkhout [4, Theorem 4.1-4] where $\pi=p^{-1/p-1}$:

Theorem 1.1. *Let h be a positive integer and let $\mathcal{M}$ be a $R(I)$-differential module such that $R(\mathcal{M},r)>r\pi^{1/p^{h-1}}$ for all $r\in I$. Then, there exists an $R(I^{p^h})$-differential module $\mathcal{N}_h$ such*

that $(\varphi^h)^*\mathcal{N}_h \cong \mathcal{M}$ and $R(\mathcal{M},r)^{p^h} = R(\mathcal{N}_h, r^{p^h})$ for any $r \in I$, and $\mathcal{N}_h$ is called a Frobenius antecedent of order h of $\mathcal{M}$.

In particular, if a $R(I)$-differential module $\mathcal{M}$ satisfies $R(\mathcal{M},r) > r\pi$ for all $r \in I$, it has a Frobenius antecedent.

2. Main theorem

In this section, I denotes an open interval in $\mathbb{R}_{>0}$ and $\mathcal{M}$ a non-Robba $\mathcal{A}(I)$-differential module associated to some matrix $G \in \mathrm{M}_\mu(\mathcal{A}(I))$.

Theorem 2.1. *Assume that the generic polygon of convergence of $\mathcal{M}$ has only one slope. Then*

$$\sup_{n\geq 0} \left\|\frac{G_n}{n!}\right\|_r R(\mathcal{M},r)^n < \infty \quad \textit{for all} \quad r \in I.$$

The proof of this theorem is based on the following lemmas:

Lemma 2.2. *Assume $R(\mathcal{M},r) > \pi r$ for all $r \in I$ and let $\mathcal{N}$ be a Frobenius antecedent of $\mathcal{M}$. Let F be an associated matrix to $\mathcal{N}$ and assume there exists a real $r_0 \in I$ such that $\sup\limits_{n\geq 0} ||\frac{F_n}{n!}||_{r_0^p} R(\mathcal{N}, r_0^p)^n < \infty$. Then $\sup\limits_{n\geq 0} ||\frac{G_n}{n!}||_{r_0} R(\mathcal{M}, r_0)^n < \infty$.*

Proof. The matrix $\mathcal{V}(z) = (\mathcal{V}_{ij}(z))_{ij} = \mathcal{V}_{F,t_{r_0}^p}(z) = \sum\limits_{n\geq 0} \frac{F_n(t_{r_0}^p)}{n!}(z - t_{r_0}^p)^n$ is the solution matrix of the differential system $\frac{d}{dz}V(z) = F(z)V(z)$ in the neighborhood of $t_{r_0}^p$ with $z = x^p$ and $\mathcal{V}(t_{r_0^p}) = \mathbb{I}_\mu$. The change of variables leads to $\frac{d}{dx}\mathcal{V}(x^p) = px^{p-1}F(x^p)\mathcal{V}(x^p)$. In addition, since $R(\mathcal{M}, r_0) > \pi r_0$, the map $x \mapsto x^p$ sends the closed disk $D(t_{r_0}, R(\mathcal{M}, r_0))$ into $D(t_{r_0}^p, R(\mathcal{M}, r_0)^p) = D(t_{r_0}^p, R(\mathcal{N}, r_0^p))$ [1, Lemma 3.1], and

$$\sup_{n\geq 0} \left\|\frac{F_n(t_{r_0}^p)}{n!}\right\| .|x^p - t_{r_0}^p|^n = \sup_{n\geq 0} \left\|\frac{F_n(t_{r_0}^p)}{n!}\right\| .|x^{p-1} + x^{p-1}t_{r_0} + \ldots + t_{r_0}^{p-1}|^n.|x - t_{r_0}|^n < \infty$$

for all $x \in D(t_{r_0}, R(\mathcal{M}, r_0))$. In the neighborhood of t_{r_o}, the matrix $\mathcal{V}_{F,t_{r_0}^p}(x^p)$ can be written as $\mathcal{V}(x^p) = \sum\limits_{n\geq 0} B_n(x - t_{r_0})^n$ where $B_n = (B_n(i,j))_{ij}$ are $\nu \times \nu$ matrices with entries un Ω. In that case, we have $\lim\limits_{n\to\infty} |B_n(i,j)|\rho^n = 0$ for any $\rho < R(\mathcal{M}, r_0)$, and therefore

$$\sup_{n\geq 0} |B_n(i,j)|\rho^n = \sup_{x\in D(t_{r_0},\rho)} |\mathcal{V}_{ij}(x^p)| \leq \sup_{z\in D(t_{r_0}^p,\rho^p)} |\mathcal{V}_{ij}(z)| = \sup_{n\geq 0} \left\|\frac{F_n(t_{r_0}^p)}{n!}\right\|\rho^{pn}. \tag{2.1}$$

Since the matrix $\mathcal{V}(z)$ is analytic and bounded in $D(t_{r_0}^p, R(\mathcal{N}, r_0^p)^-)$, there exists a positive constant $C > 0$, by [2, Proposition 2.3.3], such that

$$\sup_{n\geq 0} \left\|\frac{F_n(t_{r_0}^p)}{n!}\right\|\rho^{pn} < C \tag{2.2}$$

for any $\rho < R(\mathcal{M}, r_0)$ and close to $R(\mathcal{M}, r_0)$. Combining (2.1) and (2.2), and using again [2, Proposition 2.3.3], we find $\sup_{n\geq 0} |B_n(i,j)|R(\mathcal{M}, r_0)^n < \infty$ for all $1 \leq i, j \leq \nu$, and therefore, the matrix $\mathcal{V}(x^p)$ is analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$. In addition, since the matrix $px^{p-1}F(x^p)$ is associated to $\mathcal{M}$, then there exists an invertible matrix $H \in \mathrm{GL}_\mu(\mathcal{A}(I))$ (hence H is analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$) such that $G = H[px^{p-1}F(x^p)]$. Thus, by [2, Proposition 2.3.2], the matrix $H\mathcal{V}(x^p)$ is a solution to the system $\partial X = GX$ in the neighborhood of t_{r_0}, and moreover it is analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$. This means that $\mathcal{U}_{G,t_{r_0}}(x) = H\mathcal{V}(x^p)H(t_{r_0})^{-1}$ is also analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$. $\square$

Lemma 2.3. *The set of reals r in I for which $\sup_{n\geq 0} \|\frac{G_n}{n!}\|_r R(\mathcal{M}, r)^n < \infty$ is dense in I.*

Proof. Let J be a closed subinterval of I not reduced to a point and let ρ be a real number in the interior of J. Then, by hypothesis, $R(\mathcal{M}, \rho)/\rho < 1$ and therefore there exists an integer h such that $\pi^{1/p^{h-1}} < R(\mathcal{M}, \rho)/\rho < \pi^{1/p^h}$. Since the function $r \mapsto R(\mathcal{M}, r)$ is continuous on J, there exists an open subinterval $J' \subset J$ containing ρ such that $\pi^{1/p^{h-1}} r < R(\mathcal{M}, r) < \pi^{1/p^h} r$ for all $r \in J'$.

There are two cases to consider:

Case 1: $h \leq 0$.

Let $\dot{\mathcal{H}}(J')$ be the quotient field of $\mathcal{H}(J')$. By cyclic vector lemma, we can associate $\dot{\mathcal{H}}(J') \otimes \mathcal{M}$ to a differential equation $\Delta(\dot{\mathcal{H}}(J) \otimes \mathcal{M}) = \partial^\mu + q_1(x)\partial^{\mu-1} + \ldots + q_\mu(x)$, where $q_i \in \dot{\mathcal{H}}(J')$ for $i = 1, \ldots, \mu$. Now pick a nonempty subinterval J'' of J' such that $q_i \in \mathcal{H}(J'')$ for $i = 1, \ldots, \mu$, and let r_0 be a real number in the interval J'' and $\lambda(r_0)$ be the maximum of the p-adic absolute values of the roots of the polynomial $\Delta(\dot{\mathcal{H}}(J) \otimes \mathcal{M}) = \lambda^\mu + q_1(t_{r_0})\lambda^{\mu-1} + \ldots + q_\mu(t_{r_0})$. Since $R(\mathcal{M}, r_0) = R(\dot{\mathcal{H}}(J) \otimes \mathcal{M}, r_0) < \pi^{1/p^h} r_0 < \pi r_0$, by virtue of [6, Theorem 3.1], we have $\log(R(\mathcal{M}, r_0)) = \frac{1}{p-1} + \log(\lambda(r_0))$ and all the solutions $u_1, \ldots, u_\mu$ of $\Delta(\dot{\mathcal{H}}(J) \otimes \mathcal{M})$ in the neighborhood of t_{r_0} are analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$. Now let W be the wronskian matrix of $(u_1, \ldots, u_\mu)$. Then, W is a solution of the system

$$\partial X = A_{\Delta(\dot{\mathcal{H}}(J)\otimes\mathcal{M})} X \quad \text{where} \quad A_{\Delta(\dot{\mathcal{H}}(J)\otimes\mathcal{M})} := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -q_\mu & -q_{\mu-1} & -q_{\mu-2} & \ldots & -q_1 \end{pmatrix}.$$

Moreover, by [2, Proposition 2.3.2], the matrix W is analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$. Since G and $A_{\Delta(\dot{\mathcal{H}}(J)\otimes\mathcal{M})}$ are associated to $\mathcal{H}(J'') \otimes \mathcal{M}$, there exists a matrix $H \in \mathrm{GL}_\mu(\mathcal{H}(J''))$ such that $G = H[A_{\Delta(\dot{\mathcal{H}}(J)\otimes\mathcal{M})}]$. Since $R(\mathcal{M}, r_0) < r_0$, the matrix H is analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0)^-)$. Hence, by [2, Proposition 2.3.2], the matrix $\mathcal{U}_{G,t_{r_0}}(x) = HWH(t_{r_0})^{-1}$ is also analytic and bounded in the disk $D(t_{r_0}, R(\mathcal{M}, r_0))$. This ends the proof of the lemma in this case.

Case 2: $h > 0$.
Applying Theorem 1.1 to $\mathcal{H}(J') \otimes \mathcal{M}$, there exists a $\mathcal{H}(J'^{p^h})$-differential module $\mathcal{N}_h$ which is a Frobenius antecedent of order h of $\mathcal{H}(J') \otimes \mathcal{M}$. Moreover, $R(\mathcal{N}_h, \rho) < \pi\rho$ for all $\rho \in J'^{p^h}$. Let ${}^hF$ be an associated matrix of $\mathcal{N}_h$. Then, by case 1, there exists $r_0 \in J'$ such that ${}^hF$ is analytic and bounded in the disk $D(t_{r_0}^{p^h}, R(\mathcal{N}_h, r_0^{p^h}))$. The proof of the lemma in this case can be concluded by iteration of Lemma 2.2. □

Proof of Theorem 2.1. By hypothesis, the generic polygon of convergence of $\mathcal{M}$ has only one slope. This slope is a rational number by [5, Theorem 2]. Thus, we may assume there exist $\alpha \in \mathbb{C}_p$ and $\beta \in \mathbb{Q}$ such that $R(\mathcal{M}, r) = |\alpha| r^\beta$ for all $r \in I$.
Let now r be a real in the interior of I. Then, by Lemma 2.3, there exist two reals $r_1, r_2 \in I$ such that $r_1 < r < r_2$ and

$$\sup_{n\geq 0} \|\frac{G_n}{n!}\|_{r_1} R(\mathcal{M}, r_1)^n < \infty \quad \text{and} \quad \sup_{n\geq 0} \|\frac{G_n}{n!}\|_{r_2} R(\mathcal{M}, r_2)^n < \infty,$$

which are equivalent to

$$\sup_{n\geq 0} \|\frac{G_n}{n!}\alpha^n x^{n\beta}\|_{r_1} < \infty \quad \text{and} \quad \sup_{n\geq 0} \|\frac{G_n}{n!}\alpha^n x^{n\beta}\|_{r_2} < \infty.$$

Since all the matrices $\alpha^n x^{n\beta} G_n$ have all their entries in $\mathcal{H}[r_1, r_2]$, and for any element $f \in \mathcal{H}([r_1, r_2])$, we have $|f|_r \leq \max(|f|_{r_1}, |f|_{r_2})$, then for any integer $n \geq 0$, we have

$$\begin{aligned}
\|\frac{G_n}{n!}\|_r R(\mathcal{M}, r)^n &\leq \|\frac{G_n}{n!}\alpha^n x^{n\beta}\|_r \\
&\leq \max(\|\frac{G_n}{n!}\alpha^n x^{n\beta}\|_{r_1}, \|\frac{G_n}{n!}\alpha^n x^{n\beta}\|_{r_2}) \\
&\leq \max(\sup_{n\geq 0} \|\frac{G_n}{n!}\alpha^n x^{n\beta}\|_{r_1}, \sup_{n\geq 0} \|\frac{G_n}{n!}\alpha^n x^{n\beta}\|_{r_2}).
\end{aligned}$$

Hence,

$$\sup_{n\geq 0} \|\frac{G_n}{n!}\|_r R(\mathcal{M}, r)^n \leq \max(\sup_{n\geq 0} \|\frac{G_n}{n!}\alpha^n x^{n\beta}\|_{r_1}, \sup_{n\geq 0} \|\frac{G_n}{n!}\alpha^n x^{n\beta}\|_{r_2}) < \infty.$$

□

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